

Time Value of Money

Everything in the second half of this course is an application of one idea: money has a price that depends on when it arrives. Chapters 8 through 11 value projects, companies, bonds, and businesses, and every one of those valuations is the same arithmetic applied to a different forecast. This chapter is that arithmetic.

It is worth being clear about what is difficult here, because it is not the algebra. The formulas below are short and there are only six of them. The difficulty is the discount rate — a number that no formula in this chapter produces, that decides every answer, and that people argue about for the whole of their careers. This chapter builds the machinery and explains what the rate is made of. Chapter 10, at the end of the course, computes one.

HOW TO USE THIS CHAPTER

This chapter derives each present-value formula in general form, sets out the compounding conventions that change an answer without changing anything real, gives a discount-factor table for reading proportions at a glance, and takes the discount rate apart into the three economic components it is built from.

The videos work a single investment decision from both sides — the naive comparison and the discounted one — and then change nothing except the rate, to show the recommendation reverse.

The case applies it to a property decision where the rate is the whole argument.

6.1 Why Time Has a Price

Offered 1,000 today or 1,000 in a year, everyone takes it today. The useful question is not which you prefer but **how much less** the delayed amount would have to be before you became indifferent, because that quantity is the price of time and it can be measured.

Three separate things make the delayed amount worth less, and they are worth holding apart because they behave differently and because later chapters address them one at a time.

- **Time preference.** People prefer consumption now to consumption later, and capital has alternative uses. Even with no inflation and no risk whatever, money available today can be put to work, so parting with it requires compensation. This is the real rate.
- **Inflation.** Prices rise, so a unit of currency in a year buys less than a unit today. Chapter 5 isolated this: a completely certain promise still loses value if the price level moves.
- **Risk.** The delayed amount might not arrive in full, or at all. The less certain it is, the more compensation is required to wait for it.

Only the first two are present in a government security in your own currency, which is why such an instrument is the closest thing to a pure price of time available. Everything else adds the third component, and the size of that addition is the subject of most of the disagreements in finance.

Notice what is not on the list. **The identity of the borrower does not matter except through risk, and the use of the money does not matter at all.** A rate is a property of the cash flows being valued — their timing and their uncertainty — not of the person doing the valuing or of what they intend to buy. This is worth stating early because the opposite intuition is strong and produces a specific error: applying a firm-wide rate to a project that is far riskier or far safer than the firm as a whole.

6.2 Future Value, Present Value, and Compounding

Growing a sum forward and discounting one back are the same operation in opposite directions, and each is one line.

$$FV = PV \times (1 + r)^n$$

$$PV = FV \div (1 + r)^n$$

The exponent is what makes this interesting. Interest is earned on interest, so the effect of a rate compounds rather than accumulating, and the difference over long periods is very large. A useful shortcut is the **rule of 72**: divide 72 by the percentage rate to get the number of years a sum takes to double. At nine percent the rule gives eight years and the exact answer is 8.04, which is close enough for a conversation.

The quantity $1/(1+r)^n$ is called the **discount factor** for period n , and it is worth building an intuition for how quickly it falls, because that intuition is what stops a bad valuation before the arithmetic starts.

DISCOUNT FACTORS

The present value today of one unit received in year n

YEAR	6%	8%	10%	15%	20%
1	0.9434	0.9259	0.9091	0.8696	0.8333
2	0.8900	0.8573	0.8264	0.7561	0.6944
3	0.8396	0.7938	0.7513	0.6575	0.5787
4	0.7921	0.7350	0.6830	0.5718	0.4823
5	0.7473	0.6806	0.6209	0.4972	0.4019
10	0.5584	0.4632	0.3855	0.2472	0.1615

Read across row 10. At six percent, money a decade away is worth 56 percent of its face amount; at twenty percent it is worth 16. **The discount rate matters far more than the forecast for anything distant**, which is the single most important practical fact in this chapter. Analysts spend most of their effort refining year-eight revenue and most of the answer is decided by a rate chosen in an afternoon.

The same table explains why long-dated arguments are so often unresolvable. Two people disagreeing about whether the rate is 10 or 20 percent disagree by a factor of more than two about anything ten years out, and no amount of work on the forecast will bring them together.

IN THE VIDEOS *Videos 6.1 and 6.2 build the intuition and then the arithmetic, starting from a naive comparison that adds cash flows from different years.*

6.3 Compounding Frequency

A rate is meaningless until you know how often it is applied. A quoted twelve percent means different things depending on the compounding convention, and the differences are large enough to matter in a loan comparison.

TWELVE PERCENT, COMPOUNDED DIFFERENTLY

COMPOUNDING	FORMULA	EFFECTIVE ANNUAL RATE
Annually	$1.12 - 1$	12.000%
Semi-annually	$(1 + 0.12/2)^2 - 1$	12.360%
Quarterly	$(1 + 0.12/4)^4 - 1$	12.551%
Monthly	$(1 + 0.12/12)^{12} - 1$	12.683%
Daily	$(1 + 0.12/365)^{365} - 1$	12.747%
Continuously	$e^{0.12} - 1$	12.750%

The general conversion is worth having.

$$\text{Effective annual rate} = (1 + \text{quoted rate} / m)^m - 1$$

Three practical points follow. **Always compare instruments on their effective annual rate**, never on the quoted rate, because different lenders quote on different conventions and the comparison is otherwise meaningless. **The effect saturates**: almost all of the difference appears by monthly compounding, and daily and continuous are nearly identical. And **the gap widens with the rate** — at 12 percent the annual-to-monthly difference is 0.68 percentage points, and at 40 percent it is far larger, which matters a great deal for the short-term instruments in Chapter 4.

6.4 Net Present Value

A stream of cash flows is valued by discounting each one to today and adding the results. Subtract what it costs and the answer is the net present value.

$$\text{NPV} = \sum \text{CFT} \div (1 + r)^t - \text{Initial investment}$$

The decision rule is clean: accept a positive NPV, reject a negative one. What deserves more attention is what the number means, because "positive NPV means do it" is applied constantly by people who could not say what the quantity is.

NPV is the value created after the capital has already earned the return it could have earned elsewhere at similar risk. The required return is not something to be compared against the answer; it is already inside it. An NPV of zero does not mean the project earns nothing — it means the

project earns exactly the rate that was demanded of it, and the firm is indifferent. A positive NPV is surplus beyond that.

This explains why the rule needs no threshold. If a firm demands 18 percent and a project shows a positive NPV at 18 percent, the project clears the bar by construction. Adding a further requirement — insisting on an NPV above some figure — is double-counting the same caution twice.

The internal rate of return, in brief

The internal rate of return is the discount rate at which the net present value is exactly zero. It is useful because it can be quoted without reference to a rate, which makes it the number most often reported to people who do not want to discuss discount rates.

It also has failure modes serious enough that it should never be the primary criterion, and Chapter 8 sets them out. For now, treat it as a summary of a project's robustness: an IRR of 25 percent against a required 15 says the project survives a substantial error in the rate, which is genuinely useful information that an NPV alone does not convey.

The break-even rate

Because the answer depends on a rate nobody can pin down, the most useful output is often not a value but a threshold: **the rate at which the recommendation reverses**. A project that is worth doing at any rate below about 19 percent has been described far more usefully than one reported as being worth 40,000 at 15 percent, because the second answer conceals the assumption and the first exposes it.

This reframing does something more than presentational. It converts an argument about a number nobody can verify into a question about whether the correct rate is above or below a stated level — which is a question people with commercial judgement can actually answer.

IN THE VIDEOS Video 6.3 computes an NPV twice at two different rates, with the recommendation reversing between them.

6.5 Annuities and Perpetuities

Most real cash-flow streams are regular, and regularity collapses the summation into a closed form. Four cases cover almost everything you will meet.

THE FOUR STANDARD STREAMS

STREAM	PRESENT VALUE	CONDITIONS
Annuity	$C \times [1 - (1+r)^{-n}] / r$	n equal payments, first at the end of period 1
Annuity due	Annuity $\times (1 + r)$	Same, but the first payment is today
Perpetuity	C / r	Equal payments forever
Growing perpetuity	$C / (r - g)$	Payments growing at g forever, requires $r > g$

The annuity formula gives exactly what discounting each payment year by year and adding them up gives; it is the same sum, written in closed form. Use it written out in full, or discount year by

year in a spreadsheet. Do not reduce it to a single looked-up multiplier: a number read off a table cannot be checked, and it hides which rate and which number of periods produced it.

EIGHT ANNUAL PAYMENTS OF 100, VALUED TODAY

PERIODS	6%	8%	10%	15%
8 years	620.98	574.66	533.49	448.73

Read the six percent figure as a sentence: 800 of payments are worth 620.98 today, and the missing 179 is what waiting costs. At 15 percent the same 800 is worth 448.73, so more than three payments' worth has been consumed by delay.

Why an infinite stream has a finite value

Students find the perpetuity formula hard to believe, and the discount-factor table above is the answer. At ten percent, year 10 is worth 0.3855, year 20 is worth 0.1486, and year 50 is worth 0.0085. The payments continue forever and their contributions shrink geometrically, so the sum converges.

The practical consequence is more useful than the mathematics, and it is worth tabulating because the concentration is stronger than almost anyone guesses.

SHARE OF A PERPETUITY ARRIVING IN THE FIRST N YEARS

DISCOUNT RATE	10 YEARS	20 YEARS	30 YEARS	50 YEARS
8%	53.7%	78.5%	90.1%	97.9%
10%	61.4%	85.1%	94.3%	99.1%
12%	67.8%	89.6%	96.7%	99.7%
15%	75.3%	93.9%	98.5%	99.9%

At ten percent, 85 percent of an infinite stream of payments arrives within twenty years, and everything from year fifty onward contributes less than one percent. The far future matters far less than it feels like it should. That instinct is worth carrying into every terminal-value discussion in Chapters 8 and 9, where the temptation is to argue about year thirty — a period that, at any realistic rate, is worth a few percent of the answer.

The growing perpetuity, and two cautions

The growing perpetuity is the engine inside every company valuation in this course, and it is extraordinarily sensitive to its inputs. Take a stream of 50 next year at a 14 percent rate.

THE SENSITIVITY OF A GROWING PERPETUITY

GROWTH ASSUMPTION	DENOMINATOR	VALUE
No growth	0.14	357.1
Growing at 4%	0.10	500.0
Growing at 7%	0.07	714.3

Three percentage points of growth added 40 percent to the value, and three more added another 43. **The denominator is a small difference between two large numbers, so both are magnified**, and this single property is responsible for most of the disagreement in the valuations you will build later.

Two errors are worth naming before they can be made.

- **C is next period's cash flow, not this period's.** If the current flow is 50 and it grows at four percent, the numerator is 52 and the value is 520, not 500. The mistake costs exactly the growth rate — four percent here — on the entire valuation, and it is extremely common.
- **The rate must exceed the growth rate.** If it does not, the denominator is zero or negative and the value is infinite or nonsensical. This is not a mathematical technicality but an economic statement: nothing can grow faster than its discount rate forever, because a business compounding above the growth rate of the economy would eventually become the entire economy.

IN THE VIDEOS *Video 6.4 works an annuity, a perpetuity, and a growing perpetuity on the same stream, and shows how much three points of growth is worth.*

6.6 The Anatomy of a Discount Rate

Section 6.1 named three components. They can now be assembled, and the assembly is worth doing because it shows where every rate you will ever be quoted comes from.

THE STACK

LAYER	WHAT IT COMPENSATES FOR	WHERE IT IS OBSERVED
Real rate	Time preference and the productive use of capital	Inflation-linked government paper
Inflation compensation	The falling purchasing power of the currency	The difference between nominal and linked yields
Practical benchmark rate	Time, inflation, and any sovereign credit premium	Same-currency government securities
Plus a risk premium	The possibility that the money does not arrive	The spread over government paper

Two observations about this table matter more than the arithmetic of stacking it up.

First, **the benchmark must be currency-specific**. In a deep market, analysts use a same-currency government security as the practical benchmark because it embeds that currency's inflation expectations and removes a currency mismatch. That does not guarantee default-free status: sovereign debt can carry credit, restructuring, and liquidity risk even in local currency. Where that risk is material, Chapter 10 starts from a developed-market default-free rate and adds country risk explicitly. A rate still cannot be moved between currencies by assumption; Chapter 10 gives the conversion.

Second, **the risk premium is an observable market price, not a theoretical construct**. When a corporate bond yields four percentage points more than a government bond of the same maturity

in the same currency, that four points is what the market charges for the possibility of not being repaid. Chapter 7 reads these spreads directly off market prices. It is worth meeting the risk premium as a price before meeting it as a formula, because the formula in Chapter 10 is an attempt to estimate something that, for debt, can simply be looked up.

6.7 Why the Rate Differs Across Cash Flows

The same firm should not use one rate for everything, because the rate belongs to the cash flows rather than to the firm. Ranking streams by the rate each deserves is a judgement, and it can be made carefully without any formula.

WHAT RAISES THE RATE A STREAM DESERVES

PROPERTY	LOWER RATE	HIGHER RATE
Contractual status	Committed under a contract	Depends on future demand
Customer quality	Government or large corporate	Fragmented, unrated, or new
Sensitivity to the economy	Demand holds in a downturn	Demand amplifies the cycle
Currency	Matches the currency of the rate	Requires conversion
Position in the queue	Secured, senior	Residual, subordinated
Length and predictability	Short, well understood	Long, with a new technology or market

Applying this table is more useful than it looks. A firm asked to value a five-year contracted maintenance stream from a state utility and a five-year forecast of retail sales in a new city should not use the same rate for both, even though both sit inside the same company. **The first is close to a bond; the second is close to a venture.** Discounting them identically overvalues the risky one and undervalues the safe one, and the error is invisible in the total.

Two practical rules follow, and they will serve until Chapter 10 supplies the arithmetic.

- **Start from an observable benchmark and adjust, saying the adjustment out loud.** A rate presented as a conclusion invites argument; a rate presented as a government yield plus a stated premium for stated reasons invites a conversation about the premium, which is the productive discussion.
- **Present the answer across a range of rates.** If the recommendation holds from 12 to 20 percent, the rate does not matter and nobody needs to argue about it. If it flips at 14, then the rate is the decision, and everyone in the room should know that before they vote.

IN THE VIDEOS Videos 6.5 and 6.6 take a rate apart into its components and then rank four cash-flow streams by the rate each deserves, arguing the ranking without computing anything.

6.8 Errors Worth Naming

- **Adding cash flows from different years.** The most basic error and the one this whole chapter exists to prevent. Five years of 900 is not 4,500 of anything.

- **Using this year's flow as C in a growing perpetuity.** Costs the growth rate on the entire terminal value.
- **Mismatching the period and the rate.** A monthly stream discounted at an annual rate, or the reverse. Convert one to match the other before starting.
- **Mixing real and nominal,** from Chapter 5. Understates value by roughly the compounded inflation rate.
- **Treating the first payment as immediate when it is not,** or the reverse. An annuity due is worth $(1+r)$ times an ordinary annuity, which is a material difference at high rates.
- **Reporting a point estimate.** Every answer in this chapter is a function of a rate nobody knows. Give the break-even rate.

6.9 Reference

CHAPTER 6 • FORMULA REFERENCE

QUANTITY	FORMULA	NOTE
Future value	$PV \times (1+r)^n$	Compounding forward
Present value	$FV / (1+r)^n$	Discounting back
Discount factor	$1 / (1+r)^n$	Tabulated in section 6.2
Net present value	$\sum C F_t / (1+r)^t - \text{investment}$	Return already inside the answer
Internal rate of return	The r at which $NPV = 0$	Failure modes in Chapter 8
Annuity	$C \times [1 - (1+r)^{-n}] / r$	First payment at end of period 1
Annuity due	$\text{Annuity} \times (1+r)$	First payment today
Perpetuity	C / r	Converges because factors shrink
Growing perpetuity	$C / (r - g)$	C is next period; requires $r > g$
Effective annual rate	$(1 + r/m)^m - 1$	Compare instruments only on this
Rule of 72	$72 / \text{rate} = \text{years to double}$	Close enough for conversation

CARRY FORWARD

The rate belongs to the cash flows, not to the firm. Two streams inside one company can deserve very different rates, and using a single rate for both is a real error rather than a simplification.

The far future contributes less than it feels like it should. At ten percent, 85 percent of a perpetuity's value arrives within twenty years.

In a growing perpetuity, C is next period's flow and r must exceed g . Both errors are common and both are large.

Report the break-even rate. It converts an argument about an unverifiable number into a question someone can actually answer.